

PROPERLY COLOURED HAMILTONIAN CYCLES IN EDGE-COLOURED COMPLETE GRAPHS

ALLAN LO

*School of Mathematics, University of Birmingham,
Birmingham, B15 2TT, UK*

ABSTRACT. Let K_n^c be an edge-coloured complete graph on n vertices. Let $\Delta_{\text{mon}}(K_n^c)$ denote the maximum number of edges of the same colour incident with a vertex of K_n^c . A properly coloured cycle is a cycle such that no two adjacent edges have the same colour. In 1976, Bollobás and Erdős [6] conjectured that every K_n^c with $\Delta_{\text{mon}}(K_n^c) < \lfloor n/2 \rfloor$ contains a properly coloured Hamiltonian cycle. In this paper, we show that for any $\varepsilon > 0$ there exists an integer n_0 such that every K_n^c with $\Delta_{\text{mon}}(K_n^c) < (1/2 - \varepsilon)n$ and $n \geq n_0$ contains a properly coloured Hamiltonian cycle. This improves an result of Alon and Gutin [1]. Thereby, the conjecture of Bollobás and Erdős is true asymptotically.

1. INTRODUCTION

An *edge-colouring* c of a graph G is an assignment of colours to the edges of G . An *edge-coloured graph* is a graph G with an edge-colouring c of G . An edge-coloured graph G is *properly coloured* if no two adjacent edges of G have the same colour. If all edges have distinct colours, then G is *rainbow*. If all edges have the same colour, then G is *monochromatic*.

Let K_n^c be an edge-coloured complete graph on n vertices. Let $\Delta_{\text{mon}}(K_n^c)$ denote the maximum number of edges of the same colour incident with a vertex of K_n^c . Equivalently, $\Delta_{\text{mon}}(K_n^c) = \max \Delta(H)$ over all monochromatic subgraphs H in K_n^c . Daykin [8] asked whether there exists a constant μ such that every K_n^c with $\Delta_{\text{mon}}(K_n^c) \leq \mu n$ and $n \geq 3$ contains a properly coloured Hamiltonian cycle. This question was answered independently by Bollobás and Erdős [6] with $\mu = 1/69$, and Chen and Daykin [7] with $\mu = 1/17$. Bollobás and Erdős proposed the following conjecture.

Conjecture 1.1 (Bollobás and Erdős [6]). *If $\Delta_{\text{mon}}(K_n^c) < \lfloor n/2 \rfloor$, then K_n^c contains a properly coloured Hamiltonian cycle.*

Later, Shearer [16] showed that $\Delta_{\text{mon}}(K_n^c) \leq n/7$ is sufficient. The best known bound on $\Delta_{\text{mon}}(K_n^c)$ was given by Alon and Gutin [1] where

E-mail address: s.a.lo@bham.ac.uk.

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$\Delta_{\text{mon}}(K_n^c) < (1 - 1/\sqrt{2} - \varepsilon)n$. On the other hand, Li, Wang and Zhou [12] showed that if $\Delta_{\text{mon}}(K_n^c) < \lfloor n/2 \rfloor$, then K_n^c contains a properly coloured cycle of length at least $(n+2)/3 + 1$.

For the existence of a properly coloured Hamiltonian path, Barr [5] proved that K_n^c contains no monochromatic triangle is a sufficient condition. Note that there is no assumption on $\Delta_{\text{mon}}(K_n^c)$. A *2-factor* is a spanning 2-regular graph. Bang-Jensen, Gutin and Yeo [4] showed that if K_n^c contains a properly coloured 2-factor, then K_n^c also contains a properly coloured Hamiltonian path. This result was later improved by Feng, Giesen, Guo, Gutin, Jensen and Rafiey [9]. A graph G is said to be a *1-path-cycle* if G is a vertex-disjoint union of at most one path P and a number of cycles. Note that a spanning 1-path-cycle without any cycles is a Hamiltonian path, and a spanning 1-path-cycle without a path is a 2-factor.

Theorem 1.2 (Feng, Giesen, Guo, Gutin, Jensen and Rafiey [9]). *Let K_n^c be an edge-coloured K_n . Then K_n^c contains a spanning properly coloured 1-path-cycle if and only if K_n^c contains a properly coloured Hamiltonian path.*

For a survey regarding properly coloured subgraphs in edge-coloured graphs, we recommend Chapter 16 of [3]. In this paper, we prove that Conjecture 1.1 is true asymptotically.

Theorem 1.3. *For any $\varepsilon > 0$, there exists an integer n_0 such that every K_n^c with $n \geq n_0$ and $\Delta_{\text{mon}}(K_n^c) \leq (1/2 - \varepsilon)n$ contains a properly coloured Hamiltonian cycle.*

For an edge-coloured graph G (not necessarily complete), the *colour degree* $d^c(v)$ of a vertex v is the number of different colours of edges incident to v . The *minimum colour degree* $\delta^c(G)$ of an edge-coloured graph G is the minimum $d^c(v)$ over all vertices v in G . Li and Wang [11] proved that every edge-coloured graph G contains a properly coloured path of length $2\delta^c(G)$ or a properly coloured cycle of length at least $2\delta^c(G)/3$. In [13], the author improved their result by showing that G contains a properly coloured path of length $2\delta^c(G)$ or a properly coloured cycle of length at least $\delta^c(G) + 1$. Furthermore, in [14], the author proved that every edge-coloured graph G on n vertices with $\delta^c(G) \geq (2/3 + \varepsilon)n$ contains a properly coloured cycle of length ℓ for all $3 \leq \ell \leq n$ provided $\varepsilon > 0$ and n is large enough. Moreover, the bound on $\delta^c(G)$ is asymptotically best possible; that is, there exist edge-coloured graphs G with $\delta^c(G) = \lfloor 2|G|/3 \rfloor - 1$ without a properly coloured Hamiltonian cycle. Note that $\delta^c(K_n^c) + \Delta_{\text{mon}}(K_n^c) \leq n$. Hence, Theorem 1.3 implies the following corollary.

Corollary 1.4. *For any $\varepsilon > 0$, there exists an integer n_0 such that every K_n^c with $n \geq n_0$ and $\delta^c(K_n^c) \geq (1/2 + \varepsilon)n$ contains a properly coloured Hamiltonian cycle.*

Now we outline the proof of Theorem 1.3, which involves two main steps. In the first step, we find (by Lemma 3.1) a small ‘absorbing cycle’ C such that for any properly coloured path P with $V(C) \cap V(P) = \emptyset$ and $|P| \geq 4$, there exists a properly coloured cycle C' with $V(C') = V(P) \cup V(C)$. This step can be viewed as a properly edge-coloured version of the absorption technique

introduced by Rödl, Ruciński and Szemerédi [15]. Since the original absorption technique did not consider edge-coloured graphs, several new ideas are needed for this generalization. We believe that there is further potential for this adaptation of the absorption technique. For instance, a similar argument was also used in [14]. In the second step, we remove the vertices of the small absorbing cycle C from K_n^c and let $K_{n'}^c$ be the resulting graph. Since C is small, we may assume that $\Delta_{\text{mon}}(K_{n'}^c) \leq (1 - \varepsilon')n'$ for some small $\varepsilon' > 0$. Next, we find a properly coloured 2-factor in $K_{n'}^c$ using Lemma 4.1. Hence, Theorem 1.2 implies that there exists a properly coloured Hamiltonian path P in $K_{n'}^c$. Finally, by the ‘absorbing’ property of C , G contains a properly coloured cycle C' with $V(C') = V(P) \cup V(C) = V(K_n^c)$. Therefore, C' is a properly coloured Hamiltonian cycle as required.

The paper is organized as follows. In the next section, we set up some basic notation and give some extremal examples to show that Conjecture 1.1 is sharp. Section 3 and Section 4 will be devoted to finding a small absorbing cycle and a properly coloured 2-factor respectively. Finally, we prove Theorem 1.3 in Section 5.

2. NOTATION AND EXTREMAL EXAMPLES

Throughout this paper unless stated otherwise, c is assumed to be an edge-colouring. Hence, $c(xy)$ is the colour of the edge xy . For $v \in V(G)$, we denote by $N_G(v)$ the neighbourhood of v in G . If the graph G is clear from the context, we omit the subscript.

Given a vertex set $U \subseteq V(G)$, write $G[U]$ to be the (edge-coloured) subgraph of G induced by U . We write $G \setminus U$ for the graph obtained from G by deleting all vertices in U . For a vertex u , we sometime write u to mean the set $\{u\}$. Given a subgraph H in G , we write $G - H$ for the graph obtained from G by deleting all edges in H . For edge-disjoint graphs G and H' , we denote by $G + H'$ the union of G and H' . We write $G - H + H'$ to mean $(G - H) + H'$.

Each path P is assumed to be directed. Hence, the paths $v_1v_2 \dots v_\ell$ and $v_\ell v_{\ell-1} \dots v_1$ are considered to be different. Given a path $P = v_1v_2 \dots v_\ell$ and a vertex $x \in N(v_1) \setminus V(P)$, we define xP to be the path $xv_1v_2 \dots v_\ell$. Similarly, given vertex-disjoint paths P_1, \dots, P_s , we define $P_1 \dots P_s$ to be the concatenation of P_1, \dots, P_s .

2.1. Extremal examples. Here, we present some edge-colourings on K_n to show that Conjecture 1.1 is sharp. The first example was given by Bollobás and Erdős [6] for $n \equiv 1 \pmod{4}$.

Example 2.1. Consider $n = 4k + 1$. Let G be a $2k$ -regular graph on n vertices. Note that the complement \overline{G} of G is also a $2k$ -regular graph. Let K_n^c be obtained by colouring all edges of G red and all edges of \overline{G} blue. Note that $\Delta_{\text{mon}}(K_n^c) = 2k = \lfloor n/2 \rfloor$. However, K_n^c does not contain any properly coloured Hamiltonian cycle C_n , since the edge-chromatic number of C_n is 3 as n is odd.

For n even, Fujita and Magnant [10] showed that there exists a K_n^c with $\delta^c(K_n^c) = n/2$ with no properly coloured Hamiltonian cycle. In fact, their example also satisfies $\Delta_{\text{mon}}(K_n^c) = n/2$. Hence, Conjecture 1.1 is also sharp

for n even. The example given by Fujita and Magnant is derived from a tournament on n vertices, that is, an oriented complete graph. In the proposition below, we present a simple generalization of their construction for general oriented graphs. Given an oriented graph \vec{G} , let $d_{\vec{G}}^-(v)$ and $d_{\vec{G}}^+(v)$ be the in- and outdegree of a vertex $v \in V(\vec{G})$. Also, define the *maximum indegree* $\Delta^-(\vec{G})$ of \vec{G} to be maximum $d_{\vec{G}}^-(v)$ over all vertices $v \in V(\vec{G})$.

Proposition 2.2. *Let G be a graph. Suppose that \vec{G} is an oriented graph obtained by orienting each edge of G . Then there exists an edge-coloured graph \tilde{G} obtained by colouring each edge of G such that*

- (i) $\Delta_{\text{mon}}(\tilde{G}) = \Delta^-(\vec{G})$;
- (ii) $d_{\tilde{G}}^c(v) = d_{\vec{G}}^+(v) + \min\{1, d_{\vec{G}}^-(v)\}$ for all $v \in V(G)$;
- (iii) C is a properly coloured cycle in \tilde{G} if and only if C is a directed cycle in \vec{G} .

Proof. Let $\{c_x : x \in V(G)\}$ be a set of distinct colours. Define an edge-colouring c of G such that for every edge $xy \in E(G)$, $c(xy) = c_y$ if and only if \vec{xy} is in \vec{G} . Set \tilde{G} be the graph G with edge-colouring c . The proposition follows. \square

Let \vec{K}_{2m} be a tournament on $2m$ vertices obtained from a regular tournament T on $2m-1$ vertices after adding a directed edge from a new vertex x to every $y \in V(T)$. Note that $\Delta^-(\vec{K}_{2m}) = m$ and \vec{K}_{2m} does not contain any directed Hamiltonian cycle. Therefore, by Proposition 2.2, there exists a K_{2m}^c (corresponding to \vec{K}_{2m}) with $\Delta(K_{2m}^c) = \Delta^-(\vec{K}_{2m}) = m$ and $\delta^c(K_{2m}^c) = m$ that does not contain any properly coloured Hamiltonian cycle.

In the proposition below, we present yet another K_n^c , which also shows that Conjecture 1.1 is sharp for n even. Moreover, this construction of K_n^c can be generalized to forbid any properly coloured paths and cycles of arbitrary length. We would like to point out that, by a suitable choice of tournament, Proposition 2.2 would also yield the same result for properly coloured cycles but not for properly coloured paths.

Proposition 2.3. *Let ℓ and n be integers with $1 \leq \ell \leq n/2$. Then there exists an edge-coloured graph K_n^c on n vertices with $\Delta_{\text{mon}}(K_n^c) = n - \ell$ and $\delta^c(K_n^c) = \ell$ such that all properly coloured cycles in K_n^c have length less than 2ℓ and all properly coloured paths in K_n^c have length less than $2\ell + 1$.*

Proof. Let the vertices of K_n be $x_1, \dots, x_\ell, y_1, \dots, y_{n-\ell}$. Set $X = \{x_1, \dots, x_\ell\}$ and $Y = \{y_1, \dots, y_{n-\ell}\}$. Let $c : E(K_n) \rightarrow \mathbb{N}$ be an edge-colouring on K_n such that:

- (a) $c(x_i y_j) = i$ for all $1 \leq i \leq \ell$ and all $1 \leq j \leq n - \ell$;
- (b) $c(y_i y_j) = 1$ for all $1 \leq i < j \leq n - \ell$;
- (c) the colour $c(x_i x_j)$ is distinct not $1, \dots, \ell$ for all $1 \leq i < j \leq \ell$.

Note that $\Delta_{\text{mon}}(K_n^c) = |Y| = n - \ell$ and $\delta^c(K_n^c) = |X| = \ell$.

Suppose C is a properly coloured cycle in K_n^c . Let P_1, P_2, \dots, P_r be the paths of C induced on the vertex set Y (where P_i may consist of one vertex). Since each P_i is properly coloured, (b) implies that $1 \leq |P_i| \leq 2$. Note that after seeing one P_i we must see at least two consecutive vertices in X , so

$$|X| \geq |X \cap V(C)| \geq 2r \geq 2\lceil |Y \cap V(C)|/2 \rceil \geq |Y \cap V(C)|. \quad (2.1)$$

Therefore

$$|C| = |X \cap V(C)| + |Y \cap V(C)| \leq 2|X| = 2\ell.$$

If $|C| = 2\ell$, then we must have equality in (2.1) and so $|X| = 2r = |Y \cap V(C)|$. Hence, we must have $|P_i| = 2$ for all $i \leq r$. Thus, each P_i is an edge of colour 1 by (b). Therefore, after seeing one P_i we must see at least two vertices x_j with $j \neq 1$ by (a) and (b) before seeing another $P_{i'}$. This implies that $|X \setminus x_1| \geq 2r = |X|$, a contradiction. Hence, all properly coloured cycles in K_n^c have length less than 2ℓ . A similar argument shows that all properly coloured paths in K_n^c have length less than $2\ell + 1$. \square

3. ABSORBING CYCLE

The aim of this section is to show that there exists a small cycle C such that for given any properly coloured path P with $V(C) \cap V(P) = \emptyset$ and $|P| \geq 4$, there exists a properly coloured cycle C' with $V(C') = V(P) \cup V(C)$.

Lemma 3.1 (Absorbing cycle lemma). *Let $0 < \varepsilon < 1/2$. Then exists an integer n_0 such that whenever $n \geq n_0$ the following holds. Suppose that K_n^c is an edge-coloured K_n with $\Delta_{\text{mon}}(K_n^c) \leq (1/2 - \varepsilon)n$. Then there exists a properly coloured cycle C with $|C| \leq 2^{-5}\varepsilon^{4\varepsilon^{-2}+2}n$ such that, for any properly coloured path P in $K_n^c \setminus V(C)$ with $|P| \geq 4$, K_n^c contains a properly coloured cycle C' with $V(C') = V(C) \cup V(P)$.*

We will need the following definition.

Definition 3.2. Let x_1, x_2, y_1, y_2 be distinct vertices in $V(K_n^c)$. A path P is an absorbing path for $(x_1, x_2; y_2, y_1)$ if the following conditions hold:

- (i) $P = z_1 z_2 z_3 z_4$ is a properly coloured path of length 3;
- (ii) $V(P) \cap \{x_1, x_2, y_1, y_2\} = \emptyset$;
- (iii) both $z_1 z_2 x_1 x_2$ and $y_2 y_1 z_3 z_4$ are properly coloured paths of length 3.

Note that the ordering of $(x_1, x_2; y_2, y_1)$ is important. Given distinct vertices x_1, x_2, y_1, y_2 , let $\mathcal{L}(x_1, x_2; y_2, y_1)$ be the set of absorbing paths P for $(x_1, x_2; y_2, y_1)$. By the definition of an absorbing path, we have the following proposition.

Proposition 3.3. *Let $P' = x_1 x_2 \dots x_{\ell-1} x_\ell$ be a properly coloured path with $\ell \geq 4$. Let $P = z_1 z_2 z_3 z_4$ be an absorbing path for $(x_1, x_2; x_{\ell-1}, x_\ell)$ with $V(P) \cap V(P') = \emptyset$. Then, $z_1 z_2 x_1 x_2 \dots x_{\ell-1} x_\ell z_3 z_4$ is a properly coloured path.*

Lemma 3.1 will be proved as follows. Suppose that $\Delta_{\text{mon}}(K_n^c) \leq (1/2 - \varepsilon)n$. In the next lemma, Lemma 3.4, we show that $\mathcal{L}(x_1, x_2; y_2, y_1)$ is large for any distinct $x_1, x_2, y_1, y_2 \in V(K_n^c)$. By a simple probabilistic argument, Lemma 3.5 shows that there exists a small family \mathcal{F}' of vertex-disjoint properly coloured paths (of order 4) such that \mathcal{F}' contains at least one absorbing

path for any distinct $x_1, x_2, y_1, y_2 \in V(K_n^c)$. Finally, we join all paths in \mathcal{F}' into one short properly coloured cycle C using Lemma 3.6. Moreover, C satisfies the desired property in Lemma 3.1.

Lemma 3.4. *Let $0 < \varepsilon < 1/8$ and let $n \geq 5\varepsilon^{-1}$ be an integer. Suppose that K_n^c is an edge-coloured K_n with $\Delta_{\text{mon}}(K_n^c) \leq (1/2 - \varepsilon)n$. Then, $\mathcal{L}(x_1, x_2; y_2, y_1) \geq \varepsilon^2 n^4/4$ for all distinct vertices $x_1, x_2, y_1, y_2 \in V(K_n^c)$.*

Proof. Fix distinct vertices $x_1, x_2, y_1, y_2 \in V(K_n^c)$. Set $V' = V(K_n^c) \setminus \{x_1, x_2, y_1, y_2\}$ and $\Delta = \Delta_{\text{mon}}(K_n^c)$. We can find two distinct vertices z_1, z_2 in V' such that $z_1 z_2 x_1 x_2$ is a properly coloured path. Note that there are $(|V'| - (\Delta - 1))(|V'| - (\Delta - 1) - 1) \geq n^2/4$ choices for z_1 and z_2 . The number of vertices $z_3 \in V' \setminus \{z_1, z_2\}$ such that $c(z_3 z_2) \neq c(z_2 z_1)$ and $c(z_3 y_1) \neq c(y_1 y_2)$ is at least

$$(|V'| - 2) - 2(\Delta - 1) = |V'| - 2\Delta \geq 2\varepsilon n - 4 \geq \varepsilon n.$$

Pick one such z_3 . By a similar argument, the number of vertices $z_4 \in V' \setminus \{z_1, z_2, z_3\}$ such that $c(z_3 z_4) \neq c(z_3 y_1)$ and $c(z_3 z_4) \neq c(z_3 z_2)$ is at least εn . Pick one such z_4 . Notice that $z_1 z_2 z_3 z_4$ is an absorbing path for $(x_1, x_2; y_2, y_1)$. Furthermore, there are at least $n^2/4 \times \varepsilon n \times \varepsilon n = \varepsilon^2 n^4/4$ many choices of z_1, z_2, z_3 and z_4 . Therefore, the proof is completed. \square

The next lemma is proved by a simple probabilistic argument since each $\mathcal{L}(x_1, x_2; y_2, y_1)$ is large.

Lemma 3.5. *Let $0 < \varepsilon < 1/2$. Then there exists an integer n_0 such that whenever $n \geq n_0$ the following holds. Suppose that K_n^c is an edge-coloured K_n with $\Delta_{\text{mon}}(K_n^c) \leq (1/2 - \varepsilon)n$. Then there exists a family \mathcal{F}' of vertex-disjoint properly coloured paths of order 4 such that $|\mathcal{F}'| \leq 2^{-7}\varepsilon^2 n$ and $|\mathcal{L}(x_1, x_2; y_2, y_1) \cap \mathcal{F}'| \geq 1$ for all distinct vertices $x_1, x_2, y_1, y_2 \in V(K_n^c)$.*

Proof. Let K_n^c be an edge-coloured K_n with $n \geq n_0$ and $\Delta_{\text{mon}}(K_n^c) \leq (1/2 - \varepsilon)n$. Recall that each path is assumed to be directed. A path $z_1 z_2 z_3 z_4$ will be considered as a 4-tuple (z_1, z_2, z_3, z_4) . Choose a family \mathcal{F} of 4-tuples in $V(K_n^c)$ by selecting each of the $n!/(n-4)!$ possible 4-tuples independently at random with probability

$$p = 2^{-8}\varepsilon^2 \frac{(n-4)!}{(n-1)!} \geq 2^{-8}\varepsilon^2 n^{-3}.$$

Then, by Chernoff's bound (see e.g. [2]) with probability $1 - o(1)$ as $n \rightarrow \infty$, the family \mathcal{F} satisfies the following properties:

$$|\mathcal{F}| \leq 2p \frac{n!}{(n-4)!} = 2^{-7}\varepsilon^2 n \quad (3.1)$$

and

$$|\mathcal{L}(x_1, x_2; y_2, y_1) \cap \mathcal{F}| > \frac{p}{2} |\mathcal{L}(x_1, x_2; y_2, y_1)| \geq 2^{-11}\varepsilon^4 n \quad (3.2)$$

for all distinct vertices $x_1, x_2, y_1, y_2 \in V(K_n^c)$, where we recall Lemma 3.4 that $|\mathcal{L}(x_1, x_2; y_2, y_1)| \geq \varepsilon^2 n^4/4$. We say that two 4-tuples (a_1, a_2, a_3, a_4) and (b_1, b_2, b_3, b_4) are *intersecting* if $a_i = b_j$ for some $1 \leq i, j \leq 4$. Furthermore,

we can bound the expected number of intersecting 4-tuples in \mathcal{F} from above by

$$\frac{n!}{(n-4)!} \times 4^2 \times \frac{(n-1)!}{(n-4)!} \times p^2 = 2^{-12} \varepsilon^4 n.$$

Thus, using Markov's inequality, we derive that with probability at least $1/2$

$$\mathcal{F} \text{ contains at most } 2^{-11} \varepsilon^4 n \text{ intersecting pairs.} \quad (3.3)$$

Hence, with positive probability the family \mathcal{F} has all properties stated in (3.1), (3.2) and (3.3). By deleting all the intersecting 4-tuples and those 4-tuples not absorbing paths in such a family \mathcal{F} , we get a subfamily \mathcal{F}' consisting of pairwise disjoint 4-tuples, which satisfies

$$|\mathcal{L}(x_1, x_2; y_2, y_1) \cap \mathcal{F}'| > 2^{-11} \varepsilon^4 n - 2^{-11} \varepsilon^4 n = 0$$

for all distinct vertices $x_1, x_2, y_1, y_2 \in V(K_n^c)$. Since each 4-tuple in \mathcal{F}' is an absorbing path, \mathcal{F}' is a set of vertex-disjoint properly coloured paths of order 4. \square

As mentioned earlier, in order to prove Lemma 3.1, we join the paths in \mathcal{F}' given by Lemma 3.5 into a short properly coloured cycle. The lemma below shows that we join any two disjoint edges by a properly coloured path of constant length.

Lemma 3.6. *Let $0 < \varepsilon < 1/72$. Then there exists an integer n_0 such that whenever $n \geq n_0$ the following holds. Suppose that K_n^c is edge-coloured with $\Delta_{\text{mon}}(K_n^c) \leq (1/2 - \varepsilon)n$. Let v_1, v_2, v'_1, v'_2 be distinct vertices. Then there exists an integer $2 \leq i_0 \leq 2\varepsilon^{-2}$ such that there are at least $(\varepsilon^2 n)^{i_0}$ paths P with $|P| = i_0$ and $v_1 v_2 P v'_2 v'_1$ is a properly coloured path.*

To illustrate the idea of the proof, we consider the following simpler problem. Suppose that K_n^c with $\Delta_{\text{mon}}(K_n^c) \leq n/3 - 3$ and let $x_1, x_2, y \in V(K_n^c)$ be distinct. We claim that there exist distinct vertices w and v such that $x_1 x_2 w v y$ is a properly coloured path. (In other words, we can join an edge $x_1 x_2$ and a vertex y into a properly coloured path of order 5.) Let $V' = V(K_n^c) \setminus \{x_1, x_2, y\}$ and X be the set of vertices $w \in V'$ such that $c(x_1 x_2) \neq c(x_2 w)$. Hence, $|W| = n - 1 - \Delta_{\text{mon}}(K_n^c) - |y| \geq 2n/3$. Define an auxiliary bipartite graph H with vertex classes W and V' and edge set $E(H)$ such that for $x \in W$ and $v \in V'$, $wx \in E(H)$ if and only if $c(wv) \neq c(wx_2)$. Hence, if $wv \in E(H)$, then $x_1 x_2 w v$ is a properly coloured path. Also, every $x \in X$ has degree at least $2n/3$ in H . Hence, by an averaging argument, there exists a vertex $v \in V'$ with degree at least $4n/9$ in H . Recall that $\Delta_{\text{mon}}(K_n^c) \leq n/3 - 3 < 4n/9$. There exist distinct $w, w' \in N_H(v)$ such that $c(vw) \neq c(vw')$. Therefore, $x_1 x_2 w v y$ or $x_1 x_2 w' v y$ is a properly coloured path as claimed.

Proof of Lemma 3.6. Let K_n^c be an edge-coloured complete graph on $n \geq n_0$ vertices with $\Delta_{\text{mon}}(K_n^c) \leq (1/2 - \varepsilon)n$. Set $\Delta = \Delta_{\text{mon}}(K_n^c)$, $V' = V(K_n^c) \setminus \{v_1, v_2, v'_1, v'_2\}$ and $n' = |V'| = n - 4$. We omit floors and ceilings for clarity of presentation.

For integers $i \geq 0$, we say that a vertex $x \in V'$ is i -far from (v_1, v_2) if there exist at least $(\varepsilon^2 n)^i$ paths P with $V(P) \subseteq V' \setminus x$ and $|P| = i$ such

that v_1v_2Px is a properly coloured path. Note that any vertex $x \in V'$ with $c(xv_2) \neq c(v_1v_2)$ is 0-far. A vertex x is *strongly i -far* if for any colour c' , after removing all edges xy with $c(xy) = c'$ there still exist at least $(\varepsilon^2n)^i/2$ paths P with $V(P) \subseteq V' \setminus x$ and $|P| = i$ such that v_1v_2Px is a properly coloured path. Hence, if x is i -far but not strongly i -far, then there exists a unique colour $c_i(x)$ such that x is no longer i -far after removing all edges xy with $c(xy) = c_i(x)$. Moreover, there are at least $(\varepsilon^2n)^i/2$ paths P with $V(P) \subseteq V' \setminus x$ and $|P| = i$ such that v_1v_2Px is a properly coloured path and the edge (in P) incident with x is of colour $c_i(x)$. Note that no vertex is strongly 0-far.

For integers $i \geq 0$, let X_i be the set of vertices in V' that are i -far but not strongly i -far. Also, let Y_i be the set of vertices in V' that are strongly i -far. Note that $Y_0 = \emptyset$. Let $N' = \{w \in V' : c(wv'_2) \neq c(v'_1v'_2)\}$. If $y \in N' \cap Y_i$, then there exist at least $(\varepsilon^2n)^i/2$ paths P such that $V(P) \subseteq V' \setminus y$, $|P| = i$ and moreover $v_1v_2Pyv'_2v'_1$ is a properly coloured path. Hence, if $|Y_i \cap N'| \geq 2\varepsilon^2n$ for some $i < 2\varepsilon^{-2}$, then the lemma holds by setting $i_0 = i + 1$. Recall that $\Delta_{\text{mon}}(K_n^c) = \Delta$, so $|N'| \geq n' - \Delta$. Therefore, to prove the lemma it is enough to show that $|Y_i| \geq \Delta + 2\varepsilon^2n$ for some integer $1 \leq i < 2\varepsilon^{-1}$.

Recall that if $x \in X_i$, then there is a unique colour $c_i(x)$ such that x is no longer i -far after removing all edges xy with $c(xy) = c_i(x)$. For each integer $0 \leq i < 2\varepsilon^{-1}$, define an auxiliary bipartite graph H_i with vertex classes $X_i \cup Y_i$ and V' and edge set $E(H_i)$ such that

- (a) every vertex y in Y_i is connected every vertex in $V' \setminus y$, and
- (b) for $x \in X_i$ and $v \in V' \setminus x$, xv is an edge in H_i if and only if $c(xv) \neq c_i(x)$.

Since $\Delta_{\text{mon}}(K_n^c) = \Delta \leq (1/2 - \varepsilon)n$, each vertex $x \in X_i$ has degree at least $n' - 1 - \Delta \geq (2 + 3\varepsilon)n/4$ in H_i . Define H'_i be a subgraph of H_i such that every $x \in X_i$ has degree exactly $(2 + 3\varepsilon)n/4$ and $y \in Y_i$ has degree $n' - 1$. Thus,

$$e(H_i) \geq e(H'_i) = (2 + 3\varepsilon)n|X_i|/4 + (n' - 1)|Y_i|. \quad (3.4)$$

Since $Y_0 = \emptyset$ and X_0 is the set of vertices $x \in V'$ such that $c(v_2x) \neq c(v_1v_2)$, $|X_0| \geq n' - \Delta \geq n/2$. Thus,

$$e(H'_0) = (2 + 3\varepsilon)|X_0|n/4 \geq nn'/4.$$

Suppose that xv is an edge in H'_i with $x \in X_i$ and $v \in V'$. Note that v is in at most $in^{i-1} \leq (\varepsilon^2n)^i/4$ paths P with $|P| = i$. Since x is i -far and $c(xv) \neq c_i(x)$, there exist at least $(\varepsilon^2n)^i/4$ paths P such that $V(P) \subseteq V' \setminus v$, $|P| = i$ and v_1v_2Pxv is a properly coloured path. A similar statement also holds for edges yv in H'_i with $y \in Y_i$ and $v \in V'$. Therefore, if a vertex $v \in V'$ has degree at least $4\varepsilon^2n$ in H'_i , then v is $(i + 1)$ -far. Moreover, we conclude that if v has degree at least $\Delta + 4\varepsilon^2n$ in H'_i , then v is strongly $(i + 1)$ -far. By counting the degrees of $v \in V'$ in H'_i , we deduce that

$$\begin{aligned} e(H'_i) &\leq 4\varepsilon^2n(n' - |X_{i+1}| - |Y_{i+1}|) + (\Delta + 4\varepsilon^2n)|X_{i+1}| + (|X_i| + |Y_i|)|Y_{i+1}| \\ &\leq 4\varepsilon^2n'n + \Delta|X_{i+1}| + (n' - 1)|Y_{i+1}| \\ &\leq e(H'_{i+1}) - \varepsilon n(3|X_{i+1}|/2 - 4\varepsilon n'), \end{aligned} \quad (3.5)$$

where the last inequality is due to (3.4). Thus,

$$e(H'_{i+1}) \geq e(H'_i) + (\varepsilon n')^2 \quad \text{if } |X_{i+1}| \geq 10\varepsilon n'/3. \quad (3.6)$$

Since $e(H'_i)$ is at most n'^2 , there exists an integer $i' \leq \varepsilon^{-2}$ such that $|X_{i'+1}| < 10\varepsilon n'/3$. Assume that i' is the smallest integer such that $|X_{i'+1}| < 10\varepsilon n'/3$. Hence, $e(H'_{i'}) \geq e(H'_0)$ by (3.6). Recall that $e(H'_0) \geq nn'/4$. By (3.5),

$$\begin{aligned} nn'/4 \leq e(H'_0) &\leq e(H'_{i'}) \leq 4\varepsilon^2 n' n + \Delta |X_{i'+1}| + n' |Y_{i'+1}| \\ &\leq 2\varepsilon n' n + n' |Y_{i'+1}|. \end{aligned}$$

Hence, $|Y_{i'+1}| \geq (1/4 - 2\varepsilon)n$. Therefore, in $H'_{i'+1}$, each vertex in V' has degree at least $|Y_{i'+1}| - 1 \geq (1/4 - 2\varepsilon)n - 1 \geq 4\varepsilon^2 n$. This implies that $X_{i'+2} \cup Y_{i'+2} = V'$ and so

$$e(H'_{i'+2}) \geq (2 + 3\varepsilon)n'n/4$$

by (3.4). Let i'' be the smallest integer $i \geq i' + 2$ such that $|X_{i+1}| < 10\varepsilon n'/3$. Recall (3.6) implies that if $|X_{i+1}| \geq 10\varepsilon n'/3$, then $e(H'_{i+1}) \geq e(H'_i) + (\varepsilon n')^2$. Since $e(H'_i) \leq n'^2$, i'' exists. Moreover, $i'' \leq i' + 2 + 1/(2\varepsilon^2) < 2\varepsilon^{-2} - 3$ as $e(H'_{i'+2}) \geq (n')^2/2$. Note that $e(H'_{i''}) \geq e(H'_{i'+2})$. By (3.5), we have

$$\begin{aligned} e(H'_{i''}) &\leq e(H'_{i''}) \leq 4\varepsilon^2 n' n + \Delta |X_{i''+1}| + n' |Y_{i''+1}| \\ (2 + 3\varepsilon)n'n/4 &\leq (4\varepsilon^2 + 5\varepsilon/3)n' n + n' |Y_{i''+1}| \\ |Y_{i''+1}| &\geq (1/2 - \varepsilon + 2\varepsilon^2)n \geq \Delta + 2\varepsilon^2 n. \end{aligned}$$

This completes the proof of the lemma. \square

We are ready to prove Lemma 3.1.

Proof of Lemma 3.1. Let K_n^c be an edge-coloured complete graph on $n \geq n_0$ vertices with $\Delta_{\text{mon}}(K_n^c) \leq (1/2 - \varepsilon)n$. Set $\gamma = \varepsilon^{2\varepsilon^{-2}+2}$. Let \mathcal{F}' be the set of properly coloured paths obtained by Lemma 3.5 (taking $\varepsilon = \gamma$). Therefore,

$$\begin{aligned} |\mathcal{F}'| &\leq 2^{-7}\gamma^2 n = 2^{-7}\varepsilon^{4\varepsilon^{-2}+4}n \\ |\mathcal{L}(x_1, x_2; y_2, y_1) \cap \mathcal{F}'| &> 0 \end{aligned}$$

for all distinct vertices $x_1, x_2, y_1, y_2 \in V(K_n^c)$. Let $P = x_1 x_2 \dots x_\ell$ be a properly coloured path with $\ell \geq 4$ and $V(P) \cap V(\mathcal{F}') = \emptyset$. Pick $P' = z_1 z_2 z_3 z_4 \in \mathcal{L}(x_1, x_2; x_{\ell-1}, x_\ell) \cap \mathcal{F}'$. Since P' is an absorbing path for $(x_1, x_2; x_{\ell-1}, x_\ell)$, Proposition 3.3 implies that $z_1 z_2 P z_3 z_4$ is a properly coloured path. Note that the endedges are the same as P' . Therefore, in order to prove Lemma 3.1, it is sufficient to join the paths of \mathcal{F}' into a properly coloured cycle C such that $|C| \leq 2^{-5}\varepsilon^{4\varepsilon^{-2}+2}n$.

Let $P_1, \dots, P_{|\mathcal{F}'|}$ be the properly coloured paths in \mathcal{F}' . For each $1 \leq j \leq |\mathcal{F}'|$, we are going to find a path Q_j with $|Q_j| \leq 2\varepsilon^{-2}$ and $V(Q_j) \subseteq V(K_n^c) \setminus V(\mathcal{F}')$ such that $P_j Q_j P_{j+1}$ is a properly coloured path and $V(Q_j) \cap V(Q_{j'}) = \emptyset$ for $j \neq j'$, where we take $P_{|\mathcal{F}'|+1} = P_1$. Assume that we have already constructed Q_1, \dots, Q_{j-1} . Let $P_j = v_1 \dots v_4$ and $P_{j+1} = v'_1 \dots v'_4$. By Lemma 3.6 (taking $v_1 = v_3$, $v_2 = v_4$, $v'_1 = v'_4$ and $v'_2 = v'_3$), there exists an integer $2 \leq i_0 \leq 2\varepsilon^{-2}$ such that there are at least $(\varepsilon^2 n)^{i_0}$ paths

Q with $|Q| = i_0$ such that $v_3v_4Qv'_1v'_2$ is a properly coloured path. Set $W_j = V(\mathcal{F}') \cup \bigcup_{j' < j} V(Q_{j'})$, so

$$|W_j| = |V(\mathcal{F}')| + \sum_{j' < j} |Q_{j'}| \leq 4|\mathcal{F}'| + 2\varepsilon^{-2}(j-1) < 2\varepsilon^{-2}|\mathcal{F}'| \leq 2^{-6}\varepsilon^{4\varepsilon^{-2}+2}n.$$

Moreover, W_i intersects with at most

$$|W_j| \times i_0 n^{i_0-1} \leq 2^{-6}\varepsilon^{4\varepsilon^{-2}+2}n \times 2\varepsilon^{-2}n^{i_0-1} = 2^{-5}\varepsilon^{4\varepsilon^{-2}}n^{i_0} < (\varepsilon^2 n)^{i_0}$$

paths of order i_0 as $i_0 \leq 2\varepsilon^{-2}$. Therefore, there exists a path Q_j with $V(Q_j) \subseteq V(K_n^c) \setminus W_j$ such that $v_3v_4Q_jv'_1v'_2$ is a properly coloured path, which implies that $P_jQ_jP_{j+1}$ is a properly coloured path. Hence, we find properly coloured paths $Q_1, \dots, Q_{|\mathcal{F}'|}$. This means that K_n^c contains a properly coloured cycle C obtained by concatenating $P_1, Q_1, P_2, Q_2, \dots, Q_{|\mathcal{F}'|}$. Note that $|C| \leq (4 + 2\varepsilon^{-2})|\mathcal{F}'| \leq 2^{-5}\varepsilon^{4\varepsilon^{-2}+2}n$. This completes the proof of Lemma 3.1. \square

4. PROPERLY COLOURED 2-FACTORS

In this section, we prove the following lemma, which finds a properly coloured 2-factor in K_n^c with $\Delta_{\text{mon}}(K_n^c) < (1/2 - \varepsilon)n$.

Lemma 4.1. *Let $0 < \varepsilon < 1/8$. Then there exists an integer n_0 such that every K_n^c with $n \geq n_0$ and $\Delta_{\text{mon}}(K_n^c) \leq (1/2 - \varepsilon)n$ contains a properly coloured 2-factor.*

Before proving the lemma, we need the following notation. Let C be a directed cycle. For a vertex $v \in V(C)$, let v_+ and v_- be the successor and ancestor of v in C respectively. Further, let $c_-(v)$ and $c_+(v)$ be the colours $c(vv_-)$ and $c(vv_+)$ respectively. For distinct vertices $u, v \in V(C)$, define vC^+u to be the path $vv_+ \dots u_-u$ on C , and similarly define vC^-u .

Given an edge-coloured graph G , we denote by $\mathcal{C}_G(v)$ the set of colours incident at v in G . Equivalently, $\mathcal{C}_G(v) = \{c(vu) : u \in N_G(v)\}$. Given $x, y \in V(G)$, the *distance* $\text{dist}_G(x, y)$ in G between x and y is the minimum integer ℓ such that G contains a path (not necessarily properly coloured) of length ℓ from x to y . Note that $\text{dist}_G(x, x) = 0$ for all $x \in V(G)$. If x and y are not connected in G , then we say $\text{dist}_G(x, y) = \infty$.

Recall that a graph G is said to be a 1-path-cycle if G is a vertex-disjoint union of at most one path P and a number of cycles. We say that G is a *1-path-cycle with parameters* $(x, c_x; y, c_y)$ if G satisfies the following three properties:

- (a) G is a properly coloured 1-path-cycle;
- (b) the path $P = v_1 \dots v_\ell$ in G has length at least 1 with $v_1 = x$ and $v_\ell = y$;
- (c) $c_x = c(v_1v_2)$ and $c_y = c(v_\ell v_{\ell-1})$.

Note that x and y are the endvertices of P . Also, c_x and c_y are precisely the colours of the edges in P (and G) intersecting x and y respectively. The ordering on P is important in this definition, so ‘a 1-path-cycle with parameters $(x, c_x; y, c_y)$ ’ is different from ‘a 1-path-cycle with parameters $(y, c_y; x, c_x)$ ’, even though the underlying graphs maybe the same. Let G be a 1-path-cycle with parameters $(x, c_x; y, c_y)$ in K_n^c . For a vertex $v \in V(G) \setminus x$,

the edge xv is a *left chord* for G if $c(xv) \neq c_x$. Similarly, the edge yv is a *right chord* for G if $v \in V(G) \setminus y$ and $c(yv) \neq c_y$. A *chord* is a left or right chord.

Now we sketch the proof of Lemma 4.1. Suppose that G is a properly coloured 1-path-cycle in K_n^c with $|G|$ maximal. Further assume the G has parameters $(x, c_x; y, c_y)$. By chord rotations (defined later), we find a properly coloured 1-path-cycle G_0 with parameters $(x', c_{x'}; y', c_{y'})$ such that $c_{x'} \neq c(x'y') \neq c_{y'}$ and $V(G_0) = V(G)$. So $x'y'$ is both a left and right chord for G_0 . Hence, $P_0 + x'y'$ is a properly coloured cycle, where P_0 is the path in G_0 . This implies that $G_0 + x'y'$ is a set of properly coloured vertex-disjoint cycles. If $V(K_n^c) = V(G) = V(G_0)$, then $G_0 + x'y'$ is a properly coloured 2-factor. If $V(K_n^c) \neq V(G)$, then $G_0 + x'y'$ together with a vertex $z \in V(K_n^c) \setminus V(G)$ is a larger properly coloured 1-path-cycle contradicting the maximality of $|G|$. This proves Lemma 4.1.

In the lemma below, we show why chords are useful.

Lemma 4.2. *Let G be a 1-path-cycle with parameters $(x, c_x; y, c_y)$. Suppose that yw is a right chord for G with $w \in V(G) \setminus \{x, y\}$. Then, there exists a properly coloured 1-path-cycle G' such that the following statements hold:*

- (i) G' is a spanning subgraph of $G + yw$ containing the edge yw .
- (ii) G' has parameters $(x, c_x; y', c_{y'})$ such that $y' \in N_G(w)$, $N_{G'}(y') = \{w, w'\}$ and $c_{y'} = c(y'w') \in \mathcal{C}_G(y') \cap \mathcal{C}_G(w)$.
- (iii) Let $N_G w = \{z_1, z_2\}$. Then, G' has parameter $(x, c_x; z_1, c')$ only if $c(yw) \neq c(wz_2)$.
- (iv) For $v \in V(G)$, if $\text{dist}_G(v, z) \geq 2$ for all $z \in \{x, y, w\}$, then $N_{G'}(v) = N_G(v)$.

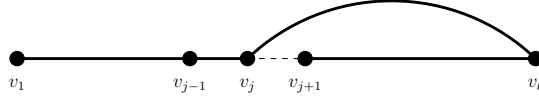
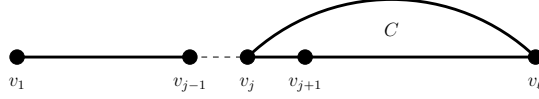
Moreover, a similar statement holds if wx is a left chord with $w \neq y$.

Proof. Let yw be a right chord for G and $P = v_1 \dots v_\ell$ be the path in G , $y = v_\ell$. First suppose that $w \notin V(P)$, so $w \in V(C)$ for some properly coloured cycle C in G . Orient C into a directed cycle such that $c(yw) \neq c_-(w)$. Observe that $P' = v_1 \dots v_\ell w C^- w_+$ is a properly coloured path. Hence, $G' = G - C - P + P'$ is a properly coloured 1-path-cycle containing yw with parameters $(x, c_x; w_+, c_+(w_+))$. Hence (i)–(iv) hold.

Next, suppose that $w \in V(P)$ and so $w = v_j$ for some $1 < j < \ell - 1$. Recall that $y = v_\ell$. Note that $c(yw) = c(v_\ell v_j) \neq c(v_j v_{j-1})$ or $c(yw) = c(v_\ell v_j) \neq c(v_j v_{j+1})$. If $c(v_\ell v_j) \neq c(v_j v_{j-1})$, then $P'' = v_1 \dots v_j v_\ell \dots v_{j+1}$ is a properly coloured path, see Figure 1. So $G' = G - P + P''$ is a properly coloured 1-path-cycle with parameters $(x, c_x; v_{j+1}, c(v_{j+1} v_{j+2}))$. If $c(v_\ell v_j) \neq c(v_j v_{j+1})$, then $C = v_j \dots v_\ell v_j$ is a properly coloured cycle, see Figure 2. Hence, $G' = G - P + C + v_1 \dots v_{j-1}$ is a properly coloured 1-path-cycle with parameters $(x, c_x; v_{j-1}, c(v_{j-1} v_{j-2}))$. Hence (i)–(iv) hold. \square

Let G , yw and G' be as defined in Lemma 4.2. We say that G' is obtained from G by a *chord rotation using the chord yw* , or a *rotation using yw* for short. Since this rotation changes the two parameters on the right, we call it a *right rotation*. Similarly, we define a *left rotation* for a left chord xw .

For the rest of this section, a chord uw is either a left or right chord (but not both) unless stated otherwise. Suppose that G' is obtained from G by a rotation using uw . Since $|\{u, w\} \cap \{x, y\}| = 1$, we can determine whether

FIGURE 1. $c(v_\ell v_j) \neq c(v_j v_{j-1})$ FIGURE 2. $c(v_\ell v_j) \neq c(v_j v_{j+1})$

the chord (and rotation) is left or right by considering $\{u, w\}$. Hence, we sometime write the chord uw as an ordered pair (u, w) with $u \in \{x, y\}$. We simply write uw for (u, w) if it is clear from the context.

Given a 1-path-cycle G with parameters $(x, c_x; y, c_y)$, we say that a 1-path-cycle G_ℓ is obtained from G by ℓ rotations using chords e_1, \dots, e_ℓ if there exist 1-path-cycles $G_1, \dots, G_{\ell-1}$ such that for each $1 \leq i \leq \ell$ the following statements hold (by taking $G_0 = G$)

- (a) e_i is a chord for G_{i-1} , and
- (b) G_i is obtained from G_{i-1} by a rotation using e_i .

The chords e_1, \dots, e_ℓ (with $e_i = (u_i, w_i)$) are said to be *spread out in G* if the distance between any two elements in $\{x, y, w_i : 1 \leq i \leq \ell\}$ is at least 5 in G . Equivalently, e_1, \dots, e_ℓ are spread out in G if $\text{dist}_G(v, v') \geq 5$ for all distinct $v, v' \in \{x, y, w_i : 1 \leq i \leq \ell\}$. The following corollary is proved by induction on ℓ together with Lemma 4.2.

Corollary 4.3. *Let G be a 1-path-cycle with parameters $(x, c_x; y, c_y)$. Let G_ℓ be a 1-path-cycle obtained from G by ℓ rotations using chords e_1, \dots, e_ℓ . Suppose that the chords e_1, \dots, e_ℓ are spread out in G and $e_i = (u_i, w_i)$ for all $1 \leq i \leq \ell$. Then, the following statements hold:*

- (i) G_ℓ has parameters $(x', c_{x'}; y', c_{y'})$ with $V(G_\ell) = V(G)$, $x', y' \in \{x, y\} \cup \bigcup_{1 \leq i \leq \ell} N_G(w_i)$, $c_{x'} \in \mathcal{C}_G(x')$ and $c_{y'} \in \mathcal{C}_G(y')$.
- (ii) For $v \in V(G)$, if $\text{dist}_G(v, u) \geq 2$ for all $u \in \{x, y\} \cup \bigcup_{1 \leq i \leq \ell} V(e_i)$, then $N_G(v) = N_{G_\ell}(v)$.
- (iii) If e_1, \dots, e_ℓ are all right chords, then $x' = x$ and $y' \in N_G(w_\ell)$.
- (iv) If e_1, \dots, e_ℓ are all left chords, then $x' \in N_G(w_\ell)$ and $y' = y$.

Proof. We proceed by induction on ℓ . The corollary is trivially true for $\ell = 0$ and so we may assume that $\ell \geq 1$. Let $G_{\ell-1}$ the 1-path-cycle with parameters $(x'', c_{x''}; y'', c_{y''})$ obtained from G by $\ell - 1$ rotations using chords $e_1, \dots, e_{\ell-1}$. Moreover, G_ℓ can be obtained from $G_{\ell-1}$ by a rotation using the chord e_ℓ . By the induction hypothesis, we have $x'', y'' \in \{x, y\} \cup \bigcup_{1 \leq i \leq \ell-1} N_G(w_i)$, $c_{x''} \in \mathcal{C}_G(x'')$ and $c_{y''} \in \mathcal{C}_G(y'')$. Since G_ℓ can be obtained from $G_{\ell-1}$ by using e_ℓ , Lemma 4.2 (ii) implies (i) holds. Similar arguments show that both (iii) and (iv) hold.

Let $v \in V(G)$ with $\text{dist}_G(v, u) \geq 2$ for all $u \in \{x, y\} \cup \bigcup_{1 \leq i \leq \ell} V(e_i)$. Note that $N_G(v) = N_{G_{\ell-1}}(v)$ by the induction hypothesis and $N_{G_{\ell-1}}(v) = N_{G_\ell}(v)$ by Lemma 4.2 (iv). Hence (ii) holds. \square

Let G be a 1-path-cycle in K_n^c of maximal order with parameters $(x, c_x; y, c_y)$. In the next lemma, we show that there exists a vertex $z \in V(G)$ and two distinct colours c^1 and c^2 such that for $i = 1, 2$ there exists 1-path-cycle G^i with parameters $(x, c_x; z, c^i)$ obtained from G by right rotations only.

Lemma 4.4. *Let $0 < \varepsilon < 1/2$. Then there exists an integer n_0 such that whenever $n \geq n_0$ the following holds. Suppose that K_n^c is an edge-coloured K_n with $\Delta_{\text{mon}}(K_n^c) \leq (1/2 - \varepsilon)n$. Let G be a properly coloured 1-path-cycle in K_n^c with $|G|$ maximal. Suppose that G has parameters $(x, c_x; y, c_y)$. Let U be a vertex set of K_n^c of size at most $\varepsilon n/8$. Then, there exist an integer $1 \leq \ell \leq \lceil 1/\log_2(1 + \varepsilon) \rceil + 1$ and a vertex $z \in V(G) \setminus U$ such that*

- (a) *For each $i = 1, 2$, there exists a 1-path-cycle G^i with parameters $(x, c_x; z, c_z^i)$ obtained from G by ℓ right rotations using chords e_1^i, \dots, e_ℓ^i and $V(G^i) = V(G)$.*
- (b) *For $i = 1, 2$, the chords e_1^i, \dots, e_ℓ^i are spread out in G and $V(e_j^i) \subseteq V(G) \setminus U$ for all $j \leq \ell$.*
- (c) $c_z^1 \neq c_z^2$.

Moreover, a similar statement for left rotations.

The key step of the proof is to define Z_ℓ to be the set of pairs (z, c_z) such that $z \in V(G) \setminus U$ and there exist chords e_1, \dots, e_ℓ such that conditions (a) and (b) are satisfied. If there exists $(z, c_z), (z, c'_z) \in Z_\ell$ with $c_z \neq c'_z$, then the lemma holds. Otherwise, we show that $|Z_{\ell+1}| \geq (1 + \varepsilon)|Z_\ell|$. Since $|Z_{\ell+1}|$ is bounded above by n , we obtain a contradiction provided ℓ is large enough.

Proof of Lemma 4.4. Let K_n^c , G and U be as defined in the lemma. For integers $\ell \geq 0$, let Z_ℓ be the set of pairs (z, c_z) such that $z \in V(G) \setminus U$ and there exists a 1-path-cycle $G_\ell^{(z, c_z)}$ with parameters $(x, c_x; z, c_z)$ obtained from G by ℓ right rotations using chords e_1, \dots, e_ℓ with $V(e_j) \subseteq V(G) \setminus U$ for all $j \leq \ell$ and the chords spread out in G . Thus, $Z_0 = \{(y, c_y)\}$. To prove the lemma, it is enough to show that there exists $z \in V(G) \setminus U$ and an integer $\ell \leq \lceil 1/\log_2(1 + \varepsilon) \rceil + 1$ such that $(z, c_z), (z, c'_z) \in Z_\ell$ with $c_z \neq c'_z$.

Suppose the lemma is false. Hence, for each integer $1 \leq \ell \leq \lceil 1/\log_2(1 + \varepsilon) \rceil + 1$, if $(z, c_z) \in Z_\ell$, then c_z is uniquely determined by z and ℓ (or else we are done). Therefore,

$$|Z_\ell| \leq n \text{ for all } 0 \leq \ell \leq \lceil 1/\log_2(1 + \varepsilon) \rceil + 1. \quad (4.1)$$

We simply write $\mathbf{z} \in Z_\ell$ for $(z, c_z) \in Z_\ell$ (since c_z is uniquely determined by z and ℓ). For each $\mathbf{z} \in Z_\ell$, we fix a set of spread out chords $e_1^{\mathbf{z}}, \dots, e_\ell^{\mathbf{z}}$ in G such that there exists a 1-path-cycle $G_\ell^{\mathbf{z}}$ with parameters $(x, c_x; z, c_z)$ obtained from G by ℓ right rotations using $e_1^{\mathbf{z}}, \dots, e_\ell^{\mathbf{z}}$ with $V(e_j^{\mathbf{z}}) \cap U = \emptyset$ for all $j \leq \ell$. We denote by $P_\ell^{\mathbf{z}}$ the path in $G_\ell^{\mathbf{z}}$. Recall that $V(G_\ell^{\mathbf{z}}) = V(G)$ by Corollary 4.3 (i) and $|G|$ is maximal. For every $v \in V(K_n^c) \setminus V(G)$, we have $c(vz) = c_z$. Otherwise, we can extend $P_\ell^{\mathbf{z}}$ enlarging the 1-path-cycle $G_\ell^{\mathbf{z}}$, which contradicts the maximality of $|G|$. Since $\Delta_{\text{mon}}(K_n^c) \leq (1/2 - \varepsilon)n$,

for each $\mathbf{z} \in Z_\ell$,

$$\begin{aligned} |\{v \in V(G) \setminus z : c(vz) \neq c_z\}| &= |\{v \in V(K_n^c) \setminus z : c(vz) \neq c_z\}| \\ &\geq n - 1 - \Delta_{\text{mon}} \geq (1/2 + \varepsilon)n - 1. \end{aligned} \quad (4.2)$$

Set $U' = U \cup \bigcup_{u \in U} N_G(u)$ and $V' = V(G) \setminus U'$. So $|U'| \leq 3|U| \leq 3\varepsilon n/8$. For each integer $1 \leq \ell \leq \lceil 1/\log_2(1 + \varepsilon) \rceil$, define an auxiliary bipartite graph H_ℓ with vertex classes Z_ℓ and V' and edge set $E(H_\ell)$ such that for $\mathbf{z} \in Z_\ell$ and $v \in V'$, $\mathbf{z}v$ is an edge in H_ℓ if and only if $v \neq z$, $c(zv) \neq c_z$ and $e_1^{\mathbf{z}}, \dots, e_\ell^{\mathbf{z}}, (z, v)$ are spread out in G . Given $1 \leq \ell \leq \lceil 1/\log_2(1 + \varepsilon) \rceil$ and $\mathbf{z} = (z, c_z) \in Z_\ell$, note that the number of vertices v such that $e_1^{\mathbf{z}}, \dots, e_\ell^{\mathbf{z}}, (z, v)$ are not spread out is at most $11(\ell + 2)$. By the definition of H_ℓ and (4.2), for each $\mathbf{z} = (z, c_z) \in Z_\ell$,

$$\begin{aligned} d_{H_\ell}(\mathbf{z}) &\geq |\{v \in V(G) : c(vz) \neq c_z\}| - |U'| - 11(\ell + 2) \\ &\geq \left(\frac{1}{2} + \varepsilon\right)n - 1 - \frac{3\varepsilon n}{8} - 11\left(\left\lceil \frac{1}{\log_2(1 + \varepsilon)} \right\rceil + 3\right) \\ &\geq (1 + \varepsilon)n/2 \end{aligned} \quad (4.3)$$

as n is large. Hence,

$$e(H_\ell) \geq \sum_{\mathbf{z} \in Z_\ell} d_{H_\ell}(\mathbf{z}) \geq (1 + \varepsilon)|Z_\ell|n/2. \quad (4.4)$$

Next we investigate how $E(H_\ell)$ and $Z_{\ell+1}$ are related. Suppose that $\mathbf{z}v$ is an edge in H_ℓ with $\mathbf{z} \in Z_\ell$ and $v \in V'$. Recall that $G_\ell^{\mathbf{z}}$ is a 1-path-cycle with parameters $(x, c_x; z, c_z)$ obtained from G by rotations using $e_1^{\mathbf{z}}, \dots, e_\ell^{\mathbf{z}}$. Note that zv is a right chord for $G_\ell^{\mathbf{z}}$ as $c(zv) \neq c_z$. By Lemma 4.2, we know that there exists a 1-path-cycle G' with parameters $(x, c_x; v', c_{v'})$ obtained from $G_\ell^{\mathbf{z}}$ by a rotation using zv . This means that G' can be obtained from G by rotations using $e_1^{\mathbf{z}}, \dots, e_\ell^{\mathbf{z}}, zv$. Since $\mathbf{z}v$ is an edge in H_ℓ , the chords $e_1^{\mathbf{z}}, \dots, e_\ell^{\mathbf{z}}, zv$ are spread out in G . Recall that $v \notin U' = U \cup \bigcup_{u \in U} N_G(u)$. Corollary 4.3 (i) implies that $v' \in N_G(v)$ and so $v' \notin U$. Therefore, $(v', c_{v'}) \in Z_{\ell+1}$. So this gives a natural map ϕ from $e(H_\ell)$ to $Z_{\ell+1}$, namely $\phi(\mathbf{z}v) = (v', c_{v'})$. Note that $N_G(v') = \{v, v''\}$ and $c_{v'} = c(vv'')$ by Lemma 4.2 (ii). Recall that if $(z', c_{z'}) \in Z_\ell$, then $c_{z'}$ are uniquely determined by $c_{z'}$ and ℓ . Therefore,

$$\text{if } \mathbf{z}v, \mathbf{z}'v' \in e(H_\ell) \text{ with } v \neq v', \text{ then } \phi(\mathbf{z}v) \neq \phi(\mathbf{z}'v'). \quad (4.5)$$

So $|Z_{\ell+1}| \geq |\bigcup_{\mathbf{z} \in Z_\ell} N_{H_\ell}(\mathbf{z})|$. Since $Z_0 = \{(y, c_y)\}$, by (4.3) we have

$$|Z_1| \geq |N_{H_0}((y, c_y))| \geq (1 + \varepsilon)n/2. \quad (4.6)$$

We edge-colour H_ℓ such that the edge $\mathbf{z}v$ in H_ℓ has colour $c(zv)$. Let X_ℓ be the set of vertices in V' that see exactly one colour in H_ℓ . Let Y_ℓ be the set of vertices in V' that see at least 2 colours in H_ℓ . Given $v \in Y_\ell$, there exist $\mathbf{z}_1, \mathbf{z}_2 \in Z_\ell$ such that $\mathbf{z}_1v, \mathbf{z}_2v \in E(H_\ell)$ and $c(\mathbf{z}_1v) \neq c(\mathbf{z}_2v)$. Let $N_G(v) = \{v_1, v_2\}$. Without loss of generality, we may assume that $c(\mathbf{z}_1v) \neq c(vv_2)$ and $c(\mathbf{z}_2v) \neq c(vv_1)$. By Lemma 4.2 (ii) and (iii), there exists a 1-path cycle G' with parameters $(x, c_x; v_1, c(vv_2))$ obtained from $G_\ell^{\mathbf{z}_1}$ by rotations using \mathbf{z}_1v . Hence, $(v_1, c(vv_2)) \in Z_{\ell+1}$ and similarly $(v_2, c(vv_1)) \in Z_{\ell+1}$. In summary, every $y \in Y_\ell$ contributes to two distinct members of $Z_{\ell+1}$ and every $x \in X_\ell$ contributes to at least one member of $Z_{\ell+1}$. Moreover,

by (4.5), all members of $Z_{\ell+1}$ derived this way are distinct. This means that for $1 \leq \ell \leq \lceil 1/\log_2(1+\varepsilon) \rceil$,

$$|Z_{\ell+1}| \geq |X_\ell| + 2|Y_\ell|. \quad (4.7)$$

Since each vertex $x \in X_\ell$ meets edges of the same colour in H_ℓ , $d_{H_\ell}(x') \leq \Delta_{\text{mon}}(K_n^c) \leq n/2$. By counting the degrees of $w \in X_\ell \cup Y_\ell \subseteq V'$ in H_ℓ , we have

$$e(H_\ell) \leq \sum_{w \in X_\ell \cup Y_\ell} d_{H_\ell}(w) \leq (|X_\ell| + 2|Y_\ell|)n/2 \leq |Z_{\ell+1}|n/2,$$

where the last inequality is due to (4.7). Together with (4.4), we have

$$|Z_{\ell+1}| \geq 2e(H_\ell)/n \geq (1+\varepsilon)|Z_\ell|$$

for all $0 \leq \ell \leq \lceil \log_2(1+\varepsilon) \rceil$. Therefore,

$$|Z_{\ell+1}| \geq (1+\varepsilon)^\ell |Z_1| \geq (1+\varepsilon)^{\ell+1} n/2$$

for all $0 \leq \ell \leq \lceil 1/\log_2(1+\varepsilon) \rceil$, where the last inequality is due to (4.6). This implies that $|Z_\ell| > n$ when $\ell = \lceil 1/\log_2(1+\varepsilon) \rceil + 1$ contradicting (4.1). \square

We are ready to prove Lemma 4.1.

Proof of Lemma 4.1. Let K_n^c be an edge-coloured K_n with $\Delta_{\text{mon}}(K_n^c) \leq (1/2 - \varepsilon)n$. Let G be a properly coloured 1-path-cycle in K_n^c with $|G|$ maximal. We may assume that G is not a 2-factor or else we are done. By applying Theorem 1.2 to $K_n^c[V(G)]$, we may assume that G is a properly coloured path. Hence, G is a 1-path-cycle with parameter $(x, c_x; y, c_y)$ with $x \neq y$. Apply Lemma 4.4 (with $U = \emptyset$) to G and obtain an integer $\ell \leq \lceil 1/\log_2(1+\varepsilon) \rceil + 1$, a vertex $z \in V(G)$ and chords $e_1^1, \dots, e_\ell^1, e_1^2, \dots, e_\ell^2$ such that for $i = 1, 2$

- (a) There exists a 1-path-cycle G_R^i with parameters $(x, c_x; z, c_z^i)$ obtained from G by ℓ right rotations using e_1^i, \dots, e_ℓ^i such that $V(G_R^i) = V(G)$.
- (b) The chords e_1^i, \dots, e_ℓ^i are spread out in G and $V(e_j^i) \subseteq V(G)$ for all $j \leq \ell$.
- (c) $c_z^1 \neq c_z^2$.

Let U be the set of vertices $u \in V(G)$ such that $\text{dist}_G(u, v) \leq 5$ for some $v \in \{x, y\} \cup \bigcup_{i,j} V(e_j^i)$. Hence $|U| \leq 11(2+4\ell) \leq \varepsilon n/8$. By the left rotation version of Lemma 4.4, there exist an integer $\ell' \leq \lceil 1/\log_2(1+\varepsilon) \rceil$, a vertex $w \in V(G) \setminus U$ and chords $f_1^1, \dots, f_{\ell'}^1, f_1^2, \dots, f_{\ell'}^2$ such that for $i = 1, 2$

- (a') There exists a 1-path-cycle G_L^i with parameters $(w, c_w^i; y, c_y)$ obtained from G by ℓ' left rotations using $f_1^i, \dots, f_{\ell'}^i$ such that $V(G_L^i) = V(G)$,
- (b') The chords $f_1^i, \dots, f_{\ell'}^i$ are spread out in G and $V(f_j^i) \subseteq V(G) \setminus U$ for all $j \leq \ell'$, and
- (c') $c_w^1 \neq c_w^2$.

By (c) and (c'), we may assume without loss of generality that $c_z^1 \neq c(zw) \neq c_w^1$. Note that $e_1^1, \dots, e_\ell^1, f_1^1, \dots, f_{\ell'}^1$ are spread out in G by (b), (b') and the definition of U .

Claim 4.5. *There exists a 1-path-cycle G_0 with parameters $(w, c_w^1; z, c_z^1)$ obtained from G by rotations using $e_1^1, \dots, e_{\ell'}^1, f_1^1, \dots, f_{\ell'}^1$. Moreover, $V(G_0) = V(G)$*

Proof of claim. We proceed by induction on ℓ' . It is trivially true for $\ell' = 0$ so we may assume that $\ell' > 0$. Let $f_{\ell'} = (u, w')$. By Corollary 4.3 (i) (taking $G_{\ell} = G_L^1$), we know that $w \in N_G(w')$. Since $w' \notin \{x, y\}$ and G is a 1-path-cycle, $d_G(w') = 2$. Let

$$N_G(w') = \{w, w''\}.$$

Since G_L^1 can be obtained from G by rotations using $f_1^1, \dots, f_{\ell'}^1$, there exists a 1-path-cycle G^\diamond with parameters $(u, c_u; y, c_y)$ obtained from G by rotations using $f_1^1, \dots, f_{\ell'-1}^1$. Furthermore, $f_{\ell'}^1$ is a left chord for G^\diamond . Since $f_l = (u, w')$ and $f_1^1, \dots, f_{\ell'}^1$ are spread out in G , $\text{dist}(w', v) > 5$ for all $v \in \{x, y\} \cup \bigcup_{j \leq \ell'} V(f_j^1)$. Hence, Corollary 4.3 (ii) implies that $N_{G^\diamond}(w') = N_G(w') = \{w, w''\}$. Since G_L^1 is obtained from G^\diamond by a right rotation using the chord $f_{\ell'}^1 = (u, w')$, Lemma 4.2 (iii) (taking $G = G^\diamond$, $G' = G_L^1$ and left chord $xw = f_{\ell'}^1$) implies that

$$c(uw') \neq c(w'w''). \quad (4.8)$$

By the induction hypothesis, there exists a 1-path-cycle G_1 with parameters $(u, c_u; z, c_z^1)$ obtained from G by rotations using $e_1^1, \dots, e_{\ell'}^1, f_1^1, \dots, f_{\ell'-1}^1$. By Corollary 4.3 (ii), $N_{G_1}(w') = N_G(w') = \{w, w''\}$. Notice that $f_{\ell'}^1$ is also a left chord for G_1 . By (4.8) and Lemma 4.2 (iii) (taking $G = G_1$ and left chord $xw = f_{\ell'}^1$), there exists a 1-path-cycle G_0 with parameters $(w, c_w^1; z, c_z^1)$ obtained from G_1 by a rotation using $f_{\ell'}^1$. This completes the proof of the claim. \square

Let G_0 be the 1-path-cycle obtained by the claim. Recall that $c_z^1 \neq c(zw) \neq c_w^1$. Hence, $G_0 + zw$ is a union of vertex-disjoint properly coloured cycles with $V(G_0 + zw) = V(G_0) = V(G)$. If $V(K_n^c) \neq V(G_0)$, then $G_0 + zw$ together with a vertex $z \in V(K_n^c) \setminus V(G)$ is also properly coloured 1-path-cycle contradicting the maximality of $|G|$. Therefore, $V(G_0) = V(K_n^c)$ implying that $G_0 + zw$ is a properly coloured 2-factor in K_n^c as required. \square

5. PROOF OF THEOREM 1.3

Let K_n^c be an edge-coloured complete graph on n vertices with $\Delta_{\text{mon}}(K_n^c) \leq (1/2 - \varepsilon)n$. Without loss of generality, we may assume that $0 < \varepsilon < 1/72$. Let C be the properly coloured cycle given by Lemma 3.1 and so $|V(C)| \leq \varepsilon''n$, where $\varepsilon'' = 2^{-5}\varepsilon^{4\varepsilon^{-2}+2}$. Let $K_{n'}^c = K_n^c \setminus V(C)$. Note that

$$\Delta_{\text{mon}}(K_{n'}^c) \leq (1/2 - \varepsilon)n \leq (1/2 - \varepsilon')n'$$

where $n' = n - |V(C)|$ and $\varepsilon' = (2\varepsilon - \varepsilon'')/(2 - 2\varepsilon'')$. There exists a properly coloured 2-factor in $K_{n'}^c$ by Lemma 4.1. Since a 2-factor is a 1-path-cycle, $K_{n'}^c$ contains a properly coloured Hamiltonian path P by Theorem 1.2. By the property of C guaranteed by Lemma 3.1, there exists a properly coloured cycle C' spanning the vertex set $V(C) \cup V(P) = V(K_n^c)$. Hence, C' is a properly coloured Hamiltonian cycle in K_n^c . \square

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